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On enlargability of infinite-dimensional Lie superalgebras

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We show that in infinite dimensions a Lie superalgebra coming from a Rogers supergroup may not come from a DeWitt one. Thus, we produce evidence that a whole class of Lie superalgebras can be enlarged only by means of the "naïve" approach to supermanifolds, which therefore cannot be merely thrown away.

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Introduction

The crucial importance of Lie superalgebras (graded Lie algebras) in theoretical physics, notably supersymmetry, is well known, and the classical survey paper by Corwin, Ne'eman and Sternberg [CNS] still remains a fascinating explanation of the inner nature of the subject and its ideology. While Lie superalgebras act on a superspace by means of infinitesimal transformations, the global supersymmetry transformations constitute objects of somewhat different—and apparently less well understood—nature, the supergroups. Our present paper deals with certain mathematical aspects of a transition from a Lie superalgebra to a supergroup.

The term "enlargability" in Lie theory means a possibility to associate a Lie group to a Lie algebra; those Lie algebras coming from Lie groups are called "enlargable Lie algebras" [E, vEK]. Similarly, one of important goals of supermanifold theory is to provide means for enlarging a Lie superalgebra, by attributing

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to it a supergroup. This is done with the purpose "to recapture explicitly the geometry implicit in the algebraic structure of Lie superalgebras" [Ba].

Basically, there are two major approaches to supermanifold and supergroup theory. One of them dates back to DeWitt [DeW] and Rogers [Ro1, Ro2, Ro3, Ro4], and within this approach, a supermanifold is constructed by patching together open subsets of graded vector spaces with the help of transition functions which are superanalytic or supersmooth in the sense that they admit a familiar superfield expansion. The other approach, developed by Berezin and Leĭtes [Ber2, BerL] and Kostant [Ko], is more abstract, and a supermanifold appears as a sheaf of graded-commutative algebras on an ordinary manifold; the superfield expansion is featured as the local tensor product decomposition of the sheaf, and superfunctions come into being as purely formal entities—sections of the sheaf, rather than usual set-theoretic mappings. The advantage of the former concept is its intuitive transparency, whereas the latter approach is backed by a huge arsenal of already existing mathematical devices from algebraic geometry and sheaf theory.

There is no unity even within each one of those approaches. In particular, the DeWitt and Rogers supermanifolds and supergroups differ in that Rogers uses a finer topology on supermanifolds for glueing them together than DeWitt does. This difference results in the fact that every DeWitt supermanifold is a Rogers supermanifold, but not *vice versa* [Ro2]. However, for most purposes both in mathematical and physical applications DeWitt supermanifolds are general enough, and it was not clear whether or not there are situations where the more general Rogers supermanifolds and supergroups apply, while DeWitt supermanifolds (supergroups) do not. The present study was motivated by this problem.

It is shown in this work that in infinite dimensions a Lie superalgebra $\mathfrak{g}$ coming from a Rogers supergroup may not come from a DeWitt one. Thus, we produce evidence that a whole class of Lie superalgebras can be enlarged only by means of the most general "naïve" approach to supermanifolds, which therefore cannot be merely thrown away.

In this paper we follow the aforementioned "naïve" approach to supermanifold theory, developed along the usual lines: ground algebra Λ, graded modules over Λ, linear superspaces as even sectors of graded modules, calculus over linear superspaces, supermanifolds, supergroups and Lie superalgebras. In order to be able to construct our examples (section 6), we need a great deal of background constructions and results belonging to "global superanalysis"; most of them can be proved by a direct analogy with similar finite-dimensional results [DeW, Ro1, Ro3, Ro4, BoG, HQR, JP2, CB1, CB2, CD, Rn, BBH, BBHP, VV, Kh, KoN, MK] and/or infinite-dimensional results in the purely even case [Bou, BS, E, CD], and this is why in most cases we merely sketch the proofs.

The term "naïve" does not mean any lack of mathematical rigour. In particular, the DeWitt approach [DeW], which we extend to an infinite-dimensional setting in this paper, in its finite-dimensional form satisfies the Rothstein axiomatics for supermanifolds [Rn] as it is presented in [BBHP], [#1] and it is "naïve" only in the sense that morphisms between supermanifolds are completely determined by underlying set-theoretic mappings.

There is one subtle point about all this generalization worth mentioning. It is known that in finite dimensions, in order to produce substantial results, one needs to consider only *free* graded modules over Λ, that is, the modules of the form $\Lambda \otimes E$, where E is a (finite-dimensional) graded linear space. Of course, the same is true in infinite dimensions as well (which is unnoticed in [Kh]), and one faces the problem of the right choice of notion of a free graded module over Λ, because the tensor product should be appropriately topologized and completed thereafter; as was shown by the author [Pe3], in the case where Λ is Banach, neither the projective nor the injective tensor product result in a satisfactory concept of a free graded Λ-module. In fact, one is bound to consider ground algebras Λ which are *nuclear* as locally convex spaces: in this case all the reasonable notions of topological tensor products coincide, and the resulting notion of a graded Λ-module behaves immaculately. One can show that there is a whole natural class of nuclear graded-commutative algebras, resulting from the concept of the free graded local Arens–Michael (GLAM) algebra over a locally convex space [Pe3, Pe4]. Here we choose as the ground algebra the DeWitt algebra, Λ_∞, which is a nuclear Fréchet graded commutative algebra with a number of fascinating properties.

Our examples occur in dimension $(\infty, 0)$ and we do not enter into a systematic investigation of supermanifolds and supergroups infinite dimensional in the odd sector; already the $(0, \infty)$-dimensional supermanifolds are still enigmatic.

In the conclusion we discuss in more technical detail the significance of our present results for the development of a unified approach to supermanifold and supergroup theory.

1. The DeWitt algebra

The basic field $\mathbb{K}$ in this paper is $\mathbb{R}$. (However, all results from sections 1–5 remain true for more general valuation fields $\mathbb{K}$, including $\mathbb{C}$ and $\mathbb{Q}_p$, and the results of section 6 make sense for $\mathbb{C}$ as well.) The term *"graded"* in this paper means *"$\mathbb{Z}_2$-graded"*. A *graded* locally convex space (LCS), E, is an LCS over the basic field $\mathbb{K}$ together with a fixed decomposition $E \cong E^0 \oplus E^1$ into a direct sum of closed linear subspaces, where E^0 is called the *even* and E^1 the

[#1] Although the papers [Rn] and [BBHP] deal with Banach ground algebras only, the results can be almost *verbatim* carried over to the case of the graded local Arens–Michael algebras studied in [Pe3, Pe4]; this case already includes the DeWitt ground algebra Λ_∞.

odd part (sector) of E. The *parity* $\tilde{x}$ of an element $x \in E^0 \cup E^1$ is defined by letting $x \in E^{\tilde{x}}, \tilde{x} \in \{0,1\} = \mathbb{Z}_2$. A *(locally convex) graded-commutative algebra* is an associative algebra carrying a structure of a graded locally convex space, $\Lambda \cong \Lambda^0 \oplus \Lambda^1$, in such a way that the operations are continuous and

$$\widetilde{xy} = \tilde{x} + \tilde{y}, \quad x, y \in \Lambda^0 \cup \Lambda^1, \tag{1}$$

$$xy = (-1)^{\tilde{x}\tilde{y}} yx, \quad x, y \in \Lambda^0 \cup \Lambda^1. \tag{2}$$

Usually a graded commutative algebra is supposed to be unital.

A *local* algebra has a unique maximal ideal (which coincides with its radical). The quotient of a local algebra Λ by its radical is isomorphic to the basic field $\mathbb{K}$. The corresponding homomorphism of augmentation $\beta_\Lambda : \Lambda \to \mathbb{K}$ is called the *number part map* [Ber2] or the *body map* [DeW]. If Λ is a Banach local algebra then β_Λ is continuous [He].

The Grassmann algebra, $\wedge(q)$, with odd generators $\xi_1, \xi_2, \ldots, \xi_q$ is the best-known example of a graded commutative algebra.

We call a graded-commutative algebra Λ *effective* if the common left annihilator of the odd sector of Λ is zero:

$$^{\perp}(\Lambda^1) \stackrel{\text{def}}{=} \{x \in \Lambda : \forall y \in \Lambda^1, xy = 0\} = (0). \tag{3}$$

In other terms, Λ is effective iff the left regular representation of Λ in $\Lambda^{(\Lambda^1)}$ is faithful (cf. [CB1, CB2, JP2, Kh, VV, Pe3]).

The DeWitt algebra, Λ_∞, is the projective limit of an inverse sequence of finite-dimensional Grassmann algebras

$$\wedge(0) \leftarrow \wedge(1) \leftarrow \wedge(2) \leftarrow \cdots \leftarrow \wedge(q) \leftarrow \wedge(q+1) \leftarrow \cdots, \tag{4}$$

where the homomorphism $\wedge(q+1) \to \wedge(q)$ sends the generators $\xi_1, \xi_2, \ldots, \xi_q$ to themselves and ξ_{q+1} to 0. In other terms, it is formed by all formal power series

$$\sum_\mu \xi^\mu \tag{5}$$

in countably infinitely many free anticommuting generators $\xi_1, \xi_2, \ldots, \xi_q, \ldots$. Here μ denotes a multiindex of the form $(\mu_1, \mu_2, \ldots, \mu_k)$, $k \in \mathbb{N}$, running over the collection of all finite subsets of the set of natural numbers $\mathbb{N}$ arranged in an increasing order; by ξ^μ one denotes the monomial $\xi_1^{\mu_1} \ldots \xi_k^{\mu_k}$ [BBH]. For every q there is a canonical unital graded algebra homomorphism $\pi_q : \Lambda_\infty \to \wedge(q)$ defined by the conditions $\pi_q(\xi_i) = \xi_i$ if $i \leq q$ and $\pi_q(\xi_i) = 0$ if $i > q$. The topology is introduced by letting a sequence (x_n) converge to an element x if and only if for every q the sequence $(\pi_q x_n)$ converges to π_n in a finite-dimensional algebra $\wedge(q)$. The algebra Λ_∞ is a complete metrizable locally convex (= Fréchet) locally multiplicatively convex [He] graded commutative algebra. It is effective [contrary to $\wedge(q)$]. We denote by I_q the kernel of π_q; it is a closed graded ideal of Λ_∞.

The algebra Λ_∞ is *nuclear* as a locally convex space, being a projective limit of a sequence of finite-dimensional normed spaces $\wedge(q)$ (see [Sca]). It means that for any locally convex space E, all the "reasonable" locally convex topologizations of the tensor product $\Lambda_\infty \otimes E$ coincide (including the projective topological tensor product, weak topological tensor product, etc.) By $\Lambda_\infty \hat{\otimes} E$, as usual, we will denote the completed projective topological tensor product.

For more on Λ_∞, see [Ber1, DeW, CB1, CB2, CD, MK, Pe3].

2. Locally convex superspaces

By a *free graded Λ_∞-module* we mean a graded topological module M over Λ_∞ of the form $M \cong \Lambda_\infty \hat{\otimes} E$ where E is a graded locally convex space. For every $q \in \mathbb{N}$, the free graded (topological) $\wedge(q)$-module $\mathrm{red}_q M \cong \wedge(q) \otimes E$ is called the *qth reduction* of M. The 0th reduction of M is just E and it is called the *body* of M; we denote $M_B = \mathrm{red}_0 M$. The canonical *reduction mappings* $\pi_q^M =_{\mathrm{def}} \pi_q \hat{\otimes} \mathrm{id}_M : M \to \mathrm{red}_q M$ are continuous and linear, and the topology of a free graded Λ_∞-module M is *projective* [Sca] with respect to the family of reduction mappings in the sense that it is the coarsest topology making the reduction mappings continuous.

Denote the kernel of π_q^M by I_q^M. These kernels—and by the same token, the reduction mappings—can be defined independently of a particular choice of a tensor product decomposition of M. Namely, I_q^M is the closed submodule of M generated by $I_q \cdot M$.

If the body space M_B is purely even [that is, $M_B^1 = (0)$], then M is called a *purely even module.*

If E is Banach (Fréchet, etc.) then we call M a free graded *Banach (Fréchet, etc.) Λ_∞*-module, although, say, M is never normable as a locally convex space.

For every two free graded Λ_∞-modules, M and N, the graded Λ_∞-module of all Λ_∞-linear mappings from M to N is denoted by $L_{\Lambda_\infty(M,N)}$. If M and N are Banach then $L_{\Lambda_\infty(M,N)}$ is endowed with the (locally convex) *topology of uniform convergence on bounded subsets,* or the *β-topology.* Its base is formed by the sets

$$[B;V] \stackrel{\mathrm{def}}{=} \{f \in L_{\Lambda_\infty(M,N)} : f(B) \subset V\},$$

where B runs over the family of all bounded subsets of M, and V over the family of all open subsets of N [Sca].

Theorem 1. *Let M and N be free graded Banach Λ_∞-modules. Then $L_{\Lambda_\infty(M,N)}$ endowed with the β-topology is a free graded Banach Λ_∞-module, the graded Banach space $L(M_B, N_B)$ with the β-topology being its body.*

Proof. We will show that the β-topology on $L_{\Lambda_\infty(M,N)}$ is projective with respect to the family of canonical mappings

$$\pi_q^L \equiv \pi_q^{L_{\Lambda_\infty(M,N)}} : L_{\Lambda_\infty(M,N)} \to L_{\wedge(q)}(\mathrm{red}_q M, \mathrm{red}_q N),$$

and since the latter modules, $L_{\wedge(q)}(\mathrm{red}_q M, \mathrm{red}_q N)$, with the β-topology are verified directly to be isomorphic to $\wedge(q) \otimes L(M_B, N_B)$, the statement follows.

Indeed, let B be bounded and V be open in Λ_∞. One can assume that for some q, $V = V + I_q^M = (\pi_q^L)^{-1}(\pi_q(V))$. This implies $[B; V] = [B + I_q^M; V]$. The latter set coincides with $(\pi_q^L)^{-1}([\pi_q^L(B); \pi_q^L(V)])$. Since the set $\pi_q^L(B)$ is bounded in $\wedge(q)$, then $[\pi_q^L(B); \pi_q^L(V)]$ is open in $L_{\wedge(q)}(\mathrm{red}_q M, \mathrm{red}_q N)$. □

A subset $U \subset M$ is called *DeWitt open* if $U = (\pi_0^M)^{-1}\pi_0^M(U)$ and $\pi_0^M(U)$ is open in $E \cong M_B$. The arising *DeWitt topology* on M is obviously non-Hausdorff. Denote for a subset $U \subset M$ by $U^\sim$ the set $(\pi_0^M)^{-1}U$; then a subset $U \subset M$ is DeWitt open iff U is open and $U = U^\sim$. In most cases we will consider the restriction of the DeWitt topology to the even sector, M^0, of M.

In contrast to the DeWitt (or *coarse*) topology on M, the ordinary topology of tensor product is usually called the *Rogers*, or *fine*, topology.

The even sector, M^0, of a free graded Λ_∞-module is referred to as a *(locally convex) topological linear superspace.* (In a terminology consistent with the existence of the change of base functor, a locally convex topological superspace is rather the *space of Λ_∞-points of a superspace* than the superspace itself.) In particular, any locally convex topological linear superspace, M^0, carries a natural structure of a locally convex topological module over the topological algebra Λ_∞^0. Morphisms between locally convex topological superspaces are the so-called superlinear mappings: a mapping $f : M^0 \to N^0$ between two locally convex superspaces is called *superlinear* if it is the restriction of a continuous even Λ_∞-module morphism $\hat{f} : M \to N$ [DeW, Ro1, Ro4, BoG, HQR, JP2, CB1, CB2, CD, Rn, BBH, BBHP, VV, Kh, KoN, MK].

Let M be a free graded topological Λ_∞-module. Denote by $M^\dagger$ a graded Λ_∞-module $L_{\Lambda_\infty^0}(M^0; \Lambda_\infty)$ endowed with the β-topology, and by $M^{\dagger\dagger}$ a graded Λ_∞-module $L_{\Lambda_\infty}(M^\dagger; \Lambda_\infty)$ (cf. [JP2]). The graded topological module $M^\dagger$ is canonically isomorphic to the free graded module $L_{\Lambda_\infty}(M; \Lambda_\infty) \cong \Lambda_\infty \hat{\otimes} M_B'$, and the body of a free graded Λ_∞-module $M^{\dagger\dagger}$ is the second Banach dual space $(M_B)''$; here the prime $'$ denotes, as usual, the strong dual of a Banach space [Sca].

The Banach superspace M^0 embeds into $M^{\dagger\dagger}$ by means of the canonical evaluation mapping κ given by $\kappa(f)(x) =_{\mathrm{def}} f(x)$ for $f \in M^\dagger$ and $x \in M^0$. We identify M^0 with its image under κ in $M^{\dagger\dagger}$.

Set

$$(M^0)^\kappa \overset{\mathrm{def}}{=} \{x \in M^{\dagger\dagger} : \forall \lambda \in (\Lambda_\infty)^{\check{x}},\ \lambda x \in M^0\}.$$

The set $(M^0)^\kappa$ forms a graded topological Λ_∞-submodule of $M^{\dagger\dagger}$, and a canonical graded topological Λ_∞-module isomorphism $\hat{\kappa} : M \cong (M^0)^\kappa$ can be established such that $\hat{\kappa}|M^0 = \kappa$.

The correspondence $M^0 \mapsto (M^0)^\kappa$ is a covariant functor from the category of locally convex superspaces and Λ_∞^0-linear mappings into the category of graded topological modules and even continuous Λ_∞-linear mappings. Since every superlinear mapping between two locally convex superspaces is in particular Λ_∞^0-linear, and the above functor is inverse to the natural defining functor of restriction $M \mapsto M^0$, $f \mapsto f|M^0$, one comes to the following result.

Theorem 2. *The categories of locally convex superspaces and superlinear mappings and free graded topological modules and even continuous Λ_∞-module homomorphisms are equivalent.* □

Corollary 1. *A mapping f between two free topological graded Λ_∞-modules is Λ_∞-linear if and only if it is Λ_∞^0-linear.* □

Corollary 2. *A mapping f between two locally convex superspaces M^0 and N_0 is superlinear if and only if it is Λ_∞^0-linear.* □

In fact, the above construction $M^0 \mapsto M^\dagger \mapsto M^{\dagger\dagger}$ makes sense for an arbitrary graded topological Λ_∞-module M; however, in this case the evaluation mapping κ, still continuous, is not necessarily a topological embedding (it may even fail to be one-to-one). This extended functor enables one to prove the following

Lemma 1. *If the even part of a graded topological Λ_∞-module N is isomorphic as a topological Λ_∞^0-module to a Banach superspace M^0, and for every $x \in N_1$ there is a $\lambda \in \Lambda_\infty^1$ with $\lambda x \neq 0$, then N is isomorphic to M.* □

A mapping f from an open subset U in a topological linear superspace M^0 to a topological linear superspace N_0 is called *superdifferentiable at a point $x \in U$* if f is Gâteaux differentiable at x and the Gâteaux differential, $D_x f$, is a superlinear mapping from M^0 to N^0 and thus can be represented (uniquely) by an element of $L^0_{\Lambda_\infty(M,N)}$. A mapping f which is superdifferentiable at every point $x \in U$ in such a way that the superdifferential mapping $x \mapsto D_x f$ from M^0 to $L^0_{\Lambda_\infty(M,N)}$ is continuous, is also called a *G^1-mapping* on U and is said to belong to the graded Λ_∞-algebra $G^1(U)$. It is clear how to define recursively G^k-mappings for all $k \in \mathbb{N}$, as well as G^∞-mappings. If M^0 and N^0 are Banach superspaces then a superdifferentiable mapping is actually Fréchet differentiable, and a G^∞ mapping is Fréchet smooth (but not *vice versa*).

A mapping f from an open subset U in a Banach superspace M^0 to a Banach superspace N_0 is called *superanalytic at a point $x \in U$* if it is analytic as a mapping between Fréchet spaces and in addition the Fréchet differential, $D_x f$, is a superlinear mapping. Every superanalytic mapping is G^∞.

All the usual properties of supersmoothness and superanalyticity [DeW, Ro1, Ro4, BoG, HQR, JP2, CB1, CB2, CD, Rn, BBH, BBHP, VV, Kh, KoN, MK, BSV1] are carried over to the Banach case.

3. Supermanifolds

Let M be a free graded Banach Λ_∞-module. A *supersmooth/superanalytic Banach supermanifold modeled over* M is a Hausdorff topological space X together with a fixed supersmooth/superanalytic *atlas*, $\mathcal{A}$, on it, that is, a family of homeomorphisms on their images, $f_\alpha : U_\alpha \to X$, where $U_\alpha \subset M^0$ are open convex neighbourhoods of the origin, such that the transition functions $f_\alpha^{-1} \circ f_\beta$ are supersmooth/superanalytic in their natural domain of definition.

If there exists an atlas such that every set $U_\alpha \equiv \operatorname{dom} f_\alpha$ is DeWitt open in M^0 then the supermanifold X is called a *DeWitt supermanifold* (cf. [BBH, RaC]).

Simplest examples of supermanifolds are the *(DeWitt) superdomains* which are just (DeWitt) open convex subsets of M^0 with their natural supersmooth/superanalytic structure.

It is clear how to define morphisms between supermanifolds.

Any supersmooth/superanalytic Banach supermanifold X carries an underlying structure of an infinite-dimensional smooth/analytic Fréchet manifold. We will denote the underlying manifold by X^0. Any morphism between supermanifolds determines a morphism between underlying Fréchet manifolds, thereby a forgetful functor from the category of supermanifolds to the category of Fréchet manifolds comes into being.

Graded derivations of the sheaf $\mathcal{S}_X$ of germs of supersmooth/superanalytic mappings $X \to \Lambda_\infty$ are called *graded vector fields* on X. Graded derivations of the stalk $\mathcal{S}_{X,x}$ are called *tangent vectors to* X *at a point* $x \in X$.

Theorem 3. *The totality of tangent vectors to a Banach supermanifold X at a point $x \in X$ forms a free graded Banach Λ_∞-module T_xX canonically isomorphic to M.*

Proof. One can assume that $X = U \subset M^0$ is a superdomain and $x = 0 \in U$. The Λ_∞^0-module M^0 is isomorphic to the even part of T_xX as a topological Λ_∞^0-module; this isomorphism is given by the map

$$M^0 \ni x \mapsto [f \mapsto \partial(f|_{\{t\cdot x : t \in \mathbb{K}\}})/\partial t \in \Lambda_\infty],$$

where the derivative is the usual derivative (in any sense) of a mapping $f|_{\{t\cdot x : t\in\mathbb{K}\}} : \mathbb{R} \to \Lambda_\infty$ at 0. Now one can apply lemma 1. □

We call T_xX the *tangent module to* X *at* x.

Theorem 4. *The even part T_xX^0 of the tangent module T_xX is canonically isomorphic as a locally convex space to the tangent space to the Fréchet manifold X^0 at the point x.*

Proof. There exists a canonical isomorphism, say j, between the tangent space to the underlying Fréchet manifold X^0 at a point x and the model locally convex space, which in this case is the underlying Fréchet space, M^0_+, of M^0 (see, e.g., [Mi]). Now take as the desired isomorphism the composition of j with the isomorphism from theorem 3. □

4. Lie superalgebras

A *Lie superalgebra* over Λ_∞ is a free graded Λ_∞-module, $\mathfrak{g}$, endowed with a graded Lie bracket, which is bi-Λ_∞-linear, anticommutative and satisfies the graded Jacobi identity. For every $q \in \mathbb{N}$ the submodule $I_q^{\mathfrak{g}}$ is a graded Lie ideal in the Lie superalgebra $\mathfrak{g}$ viewed as a Lie superalgebra over the ground field $\mathbb{K}$. The quotient Lie superalgebra $\mathrm{red}_q\,\mathfrak{g} =_{\mathrm{def}} \mathfrak{g}/I_q^{\mathfrak{g}}$ carries a natural structure of a free graded $\bigwedge(q) \equiv \Lambda_\infty/I_q$-module; in addition, the graded Lie bracket on $\mathrm{red}_q\,\mathfrak{g}$ is bi-$\bigwedge(q)$-linear. This means that the *qth reduction* $\mathrm{red}_q\,\mathfrak{g}$ of $\mathfrak{g}$ is a Lie superalgebra over the finite-dimensional Grassmann algebra $\bigwedge(q)$. As a Lie $\mathbb{K}$-superalgebra, $\mathfrak{g}$ can be represented as the projective limit $\mathfrak{g} \cong \lim_{\leftarrow} \mathrm{red}_q\,\mathfrak{g}$.

If the body Lie superalgebra $\mathfrak{g}_B$ is Banach, then we refer to $\mathfrak{g}$ as a *Banach–Lie superalgebra.* In this case, the underlying graded locally convex space of $\mathfrak{g}$ is Fréchet [non-Banach unless $\mathfrak{g} = (0)$], and all the q-reductions $\mathrm{red}_q\,\mathfrak{g}$ are Banach–Lie $\mathbb{K}$-superalgebras. In particular, for every q the even sector $\mathrm{red}_q\,\mathfrak{g}^0$ is an ordinary Banach–Lie algebra. DeWitt [DeW] calls the Lie algebras $\mathrm{red}_q\,\mathfrak{g}^0$ *q-skeletons* of $\mathfrak{g}$ and denotes them by $S_q(\mathfrak{g})$. The even sector $\mathfrak{g}^0$ is a Fréchet–Lie algebra of a particular kind, the so-called *Arens–Michael* Fréchet–Lie algebra [He], that is, it is embeddable as a closed subalgebra into the direct product of a family of Banach–Lie algebras [indeed, $\mathfrak{g}^0 \cong \lim_{\leftarrow} S_q(\mathfrak{g})$].

The change of base functor from the category of Banach–Lie $\mathbb{K}$-superalgebras to the category of Banach–Lie Λ_∞-superalgebras takes a particularily simple form: $\mathfrak{h} \mapsto \Lambda_\infty \hat{\otimes}_{\mathbb{K}} \mathfrak{h}$. DeWitt calls the Banach–Lie Λ_∞-superalgebras images of the base change functor *conventional* Lie superalgebras. Unconventional Lie superalgebras exist already in dimension $(0,1)$ [DeW].

From the point of view of deformation theory [FF], any Lie superalgebra $\mathfrak{g}$ over Λ_∞ is a *deformation* of the body Lie $\mathbb{K}$-superalgebra $\mathfrak{g}_B$ over the base $\mathrm{Spec}\,\Lambda_\infty$. (The locally ringed superspace in the sense of [Ma], $\mathrm{Spec}\,\Lambda_\infty$, is a pair consisting of a one-point topological space, $\star$, and a sheaf over it with the algebra of global sections isomorphic to Λ_∞.) Usually the deformations of Lie superalgebras are considered over Grassmann algebras $\bigwedge(q)$, and it is clear that such deformations of Banach–Lie $\mathbb{K}$-superalgebras are exhausted by all q-reductions of Banach–Lie superalgebras $\mathfrak{g}$ over Λ_∞.

Lemma 2. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Banach–Lie superalgebras over Λ_∞. If the even sectors, $\mathfrak{g}^0$ and $\mathfrak{h}^0$, are isomorphic as topological Lie algebras over Λ_∞^0, then $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic as Banach–Lie superalgebras over Λ_∞.*

Proof. The canonical isomorphism $\hat{\kappa}$ between $\mathfrak{g}$ and $\mathfrak{h}$ viewed as free graded Λ_∞-modules is proved to preserve a super Lie bracket on the odd sector. The following fact, together with the effectiveness of Λ_∞, is used: for any $x, y \in \mathfrak{g}$ and any $\lambda \in \Lambda_\infty^{\tilde{x}}$, $\mu \in \Lambda_\infty^{\tilde{y}}$, one has $\lambda\mu\hat{\kappa}([x,y]) = \hat{\kappa}([\lambda x, \mu y]) = \lambda\mu[\hat{\kappa}(x), \hat{\kappa}(y)]$. □

We find it fairer to use the term *Schur–Baker–Campbell–Hausdorff–Dynkin series*, or just *SBCHD series*, for what is usually referred to as the Hausdorff series.

Theorem 5. *Let $\mathfrak{g}$ be a Banach–Lie superalgebra over Λ_∞. The SBCHD series converges in a DeWitt open neighbourhood of zero in $\mathfrak{g}^0$ and makes it into a local analytic Lie group.*

Proof. It is sufficient to prove that for every q the series $x \cdot y \equiv H(x,y)$ converges in the skeleton Banach–Lie algebra $S_q(\mathfrak{g})^0$ as $x, y \in U^\sim$, where U is an open ball in $\mathfrak{g}^0$ of radius $\frac{3}{2}\log 2$. One can show that it is sufficient to prove the convergence in the following two special cases: (a) $H(x,\alpha)$, (b) $H(-\alpha, H(\alpha, x))$, where $\alpha \in I_0^0$ and $x \in U^\sim$. Due to the nilpotency of α [indeed, $(\mathrm{ad}_\alpha)^{2^q} = 0$], there exists a very simple estimate of the SBCHD series:

$$\| H_{r,s}(x,\alpha) \| \le C \cdot M^{r+s+1} \| x \|^s,$$

where $C = \eta \cdot \max\{\|\alpha\|, \|\alpha\|^2, \ldots, \|\alpha\|^{2^q}\}$ if $r \le 2^q$, and $C = 0$ for $r > 2^q$, and M, η are positive constants taken directly from [Bou], ch. II, §7, no. 2. This inequality assures the convergence of the series $H(x,\alpha)$. A similar estimate is true for the series $H(-\alpha, H(\alpha, x))$.

To show that the SBCHD series makes $U^\sim$ into a local analytic Lie group, it is sufficient to prove the analyticity of the emerging local group operation $(x,y) \mapsto x \cdot y$. This is done once again by reducing the consideration to the skeletons $S_q(\mathfrak{g})$. They are Banach–Lie algebras, and therefore the SBCHD series determines an analytic local group law as soon as it converges at all points of an open neighbourhood of zero. In particular, the local Lie group law in each skeleton $S_q(\mathfrak{g})$ is continuous. Analyticity of a mapping in a Fréchet space means local representability as a sum of polynomial series (which has been just demonstrated) plus continuity [BS]. The continuity follows from the fact that for every $x, y \in U^\sim$ one has $\pi_q(x \cdot y) = (\pi_q x) \cdot (\pi_q y)$, that is, the local group law in $U^\sim$ is represented as the inverse limit of a sequence of continuous mappings. □

Notice that for every $q \in \mathbb{N}$, the Lie ideal $I_q^0 \cong (I_q^{\mathfrak{g}})^0 \subset \mathfrak{g}^0$ is a subset of every DeWitt neighbourhood, V, of zero in $\mathfrak{g}^0$; this means that every I_q^0 becomes a Fréchet–Lie subgroup of the local Lie group V. We will denote the Lie ideal I_q^0 viewed as a Fréchet–Lie group by $\dot{I}_q^0$. The Lie algebra of $\dot{I}_q^0$ is canonically isomorphic to the underlying Fréchet–Lie algebra of I_q^0, and the corresponding exponential mapping is identity, $\mathbb{I}\mathrm{d} : I_q^0 \to \dot{I}_q^0$. The local Lie group quotient of V by $\dot{I}_q^0$ is isomorphic to a local Lie group attached to the Lie algebra $S_q\mathfrak{g}$.

Let M be a free graded Λ_∞-module. Then the free graded Λ_∞-module $L_{\Lambda_\infty(M,M)}$ with its topology becomes a Banach–Lie superalgebra over Λ_∞ if being endowed with the supercommutator: $[f,g] =_{\mathrm{def}} fg - (-1)^{\tilde{f}\tilde{g}} gf$. This Banach–Lie superalgebra is denoted by $\mathfrak{gl}(M)$ and called the *general linear superalgebra (of M)*. It is conventional because $\mathfrak{gl}(M) \cong \Lambda_\infty \hat{\otimes} \mathfrak{gl}(M_B)$. In a categorial approach [L, BnL] $\mathfrak{gl}(M)$ is referred to as the *superalgebra of Λ_∞-points* of a general linear superalgebra.

5. Lie supergroups

A *Banach–Lie supergroup* modeled over a free graded Λ_∞-module M is a group object in the category of Banach–Lie supermanifolds over Λ_∞. In this case, it is just a supermanifold endowed with superanalytic (or supersmooth) group operations. (The corresponding structure morphisms are fully restored from the set-theoretic mappings due to the property of Λ_∞ being effective.) *In this paper we will consider superanalytic Banach–Lie supergroups only.* If the underlying supermanifold of a Banach–Lie superalgebra G is a DeWitt supermanifold then G is called a *DeWitt* Banach–Lie supergroup. Heuristically, DeWitt supergroups may be considered as deformations over Spec Λ_∞ of Berezin–Leĭtes–Kostant supergroups (= graded Lie groups). [#2]

To every Banach–Lie supergroup, G, there is associated an underlying Fréchet–Lie group, which we will denote by G^0.

The totality of all left-invariant graded vector fields on a Banach–Lie supergroup G, endowed with the graded Lie bracket of graded vector fields, forms a Banach–Lie superalgebra which is denoted by sLie G. (As a graded Λ_∞-module, it is naturally isomorphic, by means of left translations, to the tangent module T_eG to G at e_G, and thus it is free.)

[#2] To the best of our knowledge, this idea has not been shaped as a precise mathematical result yet, and there are definitely certain topological-algebraic subtleties to be surmounted.

Theorem 6. *The even sector,* $(\mathrm{sLie}\, G)^0$*, of the Banach–Lie superalgebra* sLie G *is canonically isomorphic as a Fréchet–Lie algebra to* $\mathrm{Lie}(G^0)$.

Proof. The canonical isomorphism between the corresponding tangent spaces at e from theorem 4 preserves the Lie bracket and thus is a Lie algebra isomorphism. □

For this and other basic results on supergroups, cf. [Ber2, BerL, BnL, BSV2, CB1, CB2, DeW, Ko, L, P, Pe1, Pe3, Pe4, Ro2, Ro3, Scw, SV1, SV2].

Milnor [Mi] calls a Lie group G modeled over a locally convex space a *Baker–Campbell-Hausdorff Lie group* if the (Schur–)Baker–Campbell–Hausdorff (–Dynkin) series converges on a neighbourhood of zero, V, in the locally convex Lie algebra Lie G of G, making it into a local analytical Lie group, and in addition G possesses an exponential mapping $\exp_G : \mathrm{Lie}\, G \to G$ which is a local Lie group isomorphism. We will refer to such groups as *Schur-Baker-Campbell-Hausdorff-Dynkin* (or just *SBCHD*) Lie groups. An SBCHD Lie group is always analytical. Examples of such groups are all Banach–Lie groups and (most) groups of currents, in particular, loop groups [PrS]. However, not all Fréchet–Lie groups have the SBCHD property: the group Diff S^1 has not. Since any connected simply connected SBCHD Lie group G is *regular* [Mi], then any locally convex Lie algebra morphism from Lie G to the Lie algebra Lie H of any Lie group, H, modeled over a complete locally convex space, gives rise to a Lie group morphism $G \to H$.

Theorem 7. *Let G be a superanalytic Banach–Lie supergroup. The underlying Fréchet–Lie group G^0 is an SBCHD Lie group.*

Proof. Let a chart ϕ be defined in an open ball B in a free graded Λ_∞-module M and its values cover an open neighbourhood U of e in G. Then M, first, can be identified with the tangent module to G at e and, second, becomes a Banach–Lie superalgebra (isomorphic to the Lie superalgebra of G) in the most natural way. The uniqueness of an analytic mapping with a given set of derivatives at a point implies that for every $x, y \in B$ one has

$$H(x, -y)\phi^{-1}[\phi(x)\phi(y)^{-1}] = e$$

and therefore ϕ establishes a local Lie group isomorphism between G and an SBCHD local Lie group associated to $\mathfrak{g}^0$ (theorem 5). This means that actually ϕ is an exponential mapping from $M^0 \cong (\mathrm{sLie}\, G)^0$ to the underlying Fréchet–Lie group of G, and the conditions of Milnor's definition are fulfilled. □

Theorem 8. *Let $\mathfrak{g}$ be a Banach–Lie superalgebra over Λ_∞ such that the even sector $\mathfrak{g}^0$ is enlargable as a Fréchet–Lie algebra to a SBCHD Lie group, G. Then*

G carries the structure of a Banach–Lie supergroup with sLie $G \cong \mathfrak{g}$. *Furthermore, if the neighbourhood V in the definition of a SBCHD Lie group can be chosen DeWitt open, then G can be made into a DeWitt Banach–Lie supergroup with the property* sLie $G \cong \mathfrak{g}$.

Proof. The canonical atlas of the form $\{\phi_g\}_{g\in G}$, $\phi_q(x) =_{\rm def} \exp(gx), x \in B_{(3/2)\log 2}(0)$ is used to make G into a supermanifold, where U is a neighbourhood of e such that the restriction of $\exp_G$ to U is a diffeomorphism. Both the superanalyticity of the group operations and the isomorphism sLie $G \cong \mathfrak{g}$ are verified by direct computation based on the local isomorphism between G an a local Lie group associated to $\mathfrak{g}^0$ (theorem 5) with lemma 2 involved. □

We will say that a Banach–Lie Λ_∞-superalgebra $\mathfrak{g}$ is *(Rogers) enlargable* if it comes from a Banach–Lie supergroup, G, over Λ_∞ (that is, sLie $G \cong \mathfrak{g}$), and that $\mathfrak{g}$ is *DeWitt enlargable* if it comes from a DeWitt Banach–Lie supergroup.

Theorem 9. *Every finite-dimensional Lie superalgebra* $\mathfrak{g}$ *over* Λ_∞ *is DeWitt enlargable.*

Proof. It was proved in [Pe3] that the even sector $\mathfrak{g}^0$ of a finite-dimensional Lie superalgebra $\mathfrak{g}$ over Λ_∞ comes from an SBCHD Lie group and that the neighbourhood $V \subset \mathfrak{g}^0$ in the definition of a SBCHD Lie group can be chosen DeWitt open. □

Theorem 10. *Let* $\mathfrak{g}$ *be a Banach–Lie superalgebra. Suppose all q-skeletons are enlargable Banach–Lie algebras and let* $\exp_q : S_q(\mathfrak{g}) \to G_q$ *be the corresponding exponential mappings to the Banach–Lie groups. If there exists an open neighbourhood of zero,* $U \subset \mathfrak{g}$, *such that for every q the respriction of* $\exp_q$ *to* $\pi_q(U)$ *is one-to-one, then* $\mathfrak{g}$ *is an enlargable Lie superalgebra. If there exists a DeWitt open U with the above property then* $\mathfrak{g}$ *is DeWitt enlargable.*

Proof. Set $G =_{\rm def} \lim_{\leftarrow} G_q$ and apply theorem 8. □

Theorem 11. *A conventional purely even Banach–Lie superalgebra* $\mathfrak{g}$ *is DeWitt enlargable if and only if the body* $\mathfrak{g}_B$ *is an enlargable Banach–Lie algebra.*

Proof. The even sector $\mathfrak{g}^0$ is isomorphic to the semidirect product $\mathfrak{g}^0_B \ltimes_\tau I^{\mathfrak{g}}_0$. The Lie algebra action $\tau : \mathfrak{g}^0_B \to \operatorname{Der} I^{\mathfrak{g}}_0$ gives rise to a smooth action $\dot{\tau} : G_0 \to \operatorname{Aut} \dot{I}^{\mathfrak{g}}_0$, where G_0 is a connected simply connected Banach–Lie group attached to $\mathfrak{g}^0_B$. (This is proved at the level of q-reductions of the Lie ideal $I^{\mathfrak{g}}_0$, by means of

Banach–Lie theory [Bou].) Now the desired SBCDH Fréchet–Lie group assigned to $\mathfrak{g}^0$ is the semidirect product $G^0 \cong G_0 \ltimes_{\tau} \dot{I}_0^{\mathfrak{g}}$, the exponential map being defined by $\exp_{G^0}(x,y) =_{\mathrm{def}} (\exp_{G_0}(x), y)$. □

In particular, the following is true.

Theorem 12. *The general linear superalgebra* $\mathfrak{gl}(M)$ *is DeWitt enlargable for an arbitrary free graded* Λ_∞*-module* M*, and the corresponding Banach–Lie supergroup is the general linear supergroup* $GL(M)$*.* □

Theorem 13. *A Banach–Lie superalgebra* $\mathfrak{g}$*, admitting a continuous graded Lie monomorphism into an enlargable Banach–Lie superalgebra* $\mathfrak{h}$*, is enlargable. Moreover, if* $\mathfrak{h}$ *is DeWitt enlargable, then so is* $\mathfrak{g}$*.*

Proof. Based on the theorem on extension of analytic structure [Ṡ] and theorem 8. □

A Lie superalgebra $\mathfrak{g}$ is *centerless* if its *supercenter*, $\mathfrak{sz}(\mathfrak{g}) = \{x \in \mathfrak{g} : \forall y \in \mathfrak{g}, [x,y] = 0\}$, is zero. Although it is known that every centerless Banach–Lie algebra is enlargable [vEK], the even sector $\mathfrak{g}^0$ of a centerless Banach–Lie superalgebra $\mathfrak{g}$ is not necessarily a centerless Lie algebra. Therefore, the following result is of interest.

Theorem 14. *Every centerless Banach–Lie superalgebra* $\mathfrak{g}$ *is DeWitt enlargable.*

Proof. The proof is similar to the proof of the corresponding purely even result and it is based on the fact that $\mathfrak{g}$ admits a faithful linear representation—which is, of course, the adjoint representation $x \mapsto \mathrm{ad}_x$—and thus a continuous monomorphism from $\mathfrak{g}$ to the general linear superalgebra $\mathfrak{gl}(\mathfrak{g}_+)$ comes into being, where $\mathfrak{g}_+$ stands for the underlying free graded Λ_∞-module of $\mathfrak{g}$. Now we use theorems 8 and 13. □

6. Examples

Fix an enlargable Banach–Lie algebra $\mathfrak{k}$ and a topological two-cocycle $\eta \in H^2(\mathfrak{k};\mathbb{R})$ such that the period group [vEK] of this cocycle, $\mathrm{Per}(\eta)$, is an infinite cyclic subgroup of $\mathbb{R}$; we can assume that $\mathrm{Per}(\eta) \cong \mathbb{Z}$. In other words, this means that a one-dimensional central extension of $\mathfrak{k}$ by means of the cocycle η, a Banach–Lie algebra $\mathfrak{l} \cong \mathfrak{k} \times_\eta \mathbb{R}$, is enlargable, and the connected simply connected Banach–Lie group I corresponding to $\mathfrak{k} \times_\eta \mathbb{R}$ is a central extension of the connected simply connected Banach–Lie group K corresponding to $\mathfrak{k}$ by means of the one-dimensional toroidal group $\mathrm{U}(1) \cong \mathbb{R}/\mathbb{Z}$:

$$e \to \mathrm{U}(1) \to I \to K \to e\,.$$

Different examples of this kind can be found in [vEK, LaT, Bou, E, PrS].

Recall that $\xi_1, \xi_2, \ldots, \xi_q, \ldots$ denote a fixed system of topologically free odd generators for the algebra Λ_∞.

Example 1. *Enlargable Banach–Lie superalgebra which is non DeWitt enlargable.* Put $\mathfrak{h} =_{\text{def}} \Lambda_\infty \hat{\otimes} \mathfrak{k}$ and let $\theta = \xi_1\xi_2\eta \in H^2_{\Lambda_\infty}(\mathfrak{h}; \Lambda_\infty)$ be a bi-Λ_∞-linear even topological two-cocycle on $\mathfrak{h}$ with coefficients in Λ_∞. Let $\mathfrak{g} =_{\text{def}} \Lambda_\infty \times_\theta \mathfrak{h}$ be a Banach–Lie superalgebra one-dimensional central extension (over Λ_∞) of $\mathfrak{h}$ by means of the two-cocycle θ.

For every $q \in \mathbb{N}$, the q-skeleton algebra $S_q\mathfrak{g}$ is enlargable and the restriction of the corresponding exponential mapping $\exp_q : S_q\mathfrak{g} \to G_q$ to the open unit ball is one-to-one. Indeed, the period group of the extension

$$0 \to \Lambda^0_\infty \to S_q(\mathfrak{g}) \to S_q(\mathfrak{h}) \cong \wedge(q) \otimes \mathfrak{k} \to 0$$

is isomorphic to $\xi_1\xi_2\mathbb{Z}$, and therefore the corresponding Banach–Lie group extension is well defined:

$$e \to \Lambda^0_\infty/\xi_1\xi_2\mathbb{Z} \to G_q \to S_q(I) \to e\,,$$

where by I_q we denote the q-skeleton of a connected simply connected Fréchet–Lie group associated to the conventional Lie algebra $\mathfrak{h}^0$.

Now an application of theorem 10 implies that $\mathfrak{g}$ is an enlargable Banach–Lie superalgebra.

Suppose $\mathfrak{g}$ is a DeWitt enlargable Banach–Lie superalgebra. Then there exists an analytic DeWitt Lie supergroup G attached to $\mathfrak{g}$. According to theorems 6 and 7, the underlying Fréchet–Lie group G^0 of G is a SBCHD Fréchet–Lie group associated to the even sector $\mathfrak{g}^0$. The mapping $i : (r, x) \mapsto (r\xi_1\xi_2, 1_{\Lambda_\infty} \otimes x)$ determines an embedding of the Banach–Lie algebra $\mathfrak{l}$ into the Fréchet–Lie algebra $\mathfrak{g}$ as a (closed) locally convex Lie subalgebra. Therefore, there exists a Lie group morphism $\hat{i}$ from I to G^0 such that the corresponding exponential mappings commute: $\exp_{G^0} \circ i = \hat{i} \circ \exp_I$. However, for an element $y =_{\text{def}} (1,0) \in \mathfrak{l}$ one has $\exp_{G^0} \circ i(y) = \exp_{G^0}(\xi_1\xi_2, e) \neq 0$, because of injectivity of $\exp_{G^0}$ along the "soul direction," while $\hat{i} \circ \exp_I(y) = \hat{i}(e_I) = e_{G^0}$. This contradiction means that $\mathfrak{g}$ is not DeWitt enlargable.

Example 2. *Non-enlargable Banach–Lie superalgebra of which all q-skeletons are enlargable Banach–Lie algebras.* Define a Banach–Lie algebra $\mathfrak{m}$ as an l_1-type sum of countably many copies of the algebra $\mathfrak{k}$, that is, $\mathfrak{m}$ is isomorhic to the completion of the Lie algebra $\oplus_{i\in\mathbb{N}}\mathfrak{k}_{(i)}$ endowed with the norm

$$\|(x_i)_{i=1}^{\infty}\|_{\oplus_{i=1}^{\infty}\mathfrak{k}_{(i)}} \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \|x_i\|_{\mathfrak{k}}\,.$$

Define a topological bi-Λ_∞-linear even two-cocycle θ on $\mathfrak{m}$ with coefficients in Λ_∞ by

$$\theta[(x_i)_{i=1}^{\infty}, (y_j)_{j=1}^{\infty}] \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \xi_n\xi_{n+1}\eta(x_n, y_n)\,.$$

Set $\mathfrak{h} =_{\text{def}} \Lambda_\infty \hat{\otimes}\, \mathfrak{m}$ and let $\mathfrak{g} =_{\text{def}} \Lambda_\infty \times_\theta \mathfrak{h}$ be a Banach–Lie superalgebra one-dimensional central extension (over Λ_∞) of $\mathfrak{m}$ by means of the two-cocycle θ.

For every $q \in \mathbb{N}$, the q-skeleton Banach–Lie algebra $S_q(\mathfrak{g})$ is enlargable. Indeed, the period group of the extension

$$0 \to \Lambda_\infty^0 \to S_q(\mathfrak{g}) \to S_q(\mathfrak{m}) \to 0$$

is isomorphic to the discrete subgroup $(\sum_{n=1}^{q} \xi_n\xi_{n+1})\mathbb{Z}$, and therefore the corresponding Banach–Lie group extension is well defined.

At the same time, the Banach–Lie superalgebra $\mathfrak{g}$ is non enlargable. Indeed, for every $n \in \mathbb{N}$, there exists an embedding

$$i_k : (r, x) \mapsto (r\xi_n\xi_{n+1}, 1_{\Lambda_\infty} \otimes x)$$

of the Banach–Lie algebra $\mathfrak{l}$ in $\mathfrak{g}^0$ as a closed locally convex subalgebra, and arguments similar to those used in example 1 show that if $\mathfrak{g}^0$ were enlargable then the exponential mapping on $\mathfrak{g}^0$ would send all elements of the form $(\xi_n\xi_{n+1}, 0)$ to the identity of the corresponding Fréchet–Lie group. Since elements of the form $(\xi_n\xi_{n+1}, 0)$ are to be found in every neighbourhood of zero in $\mathfrak{g}^0$, then the exponential map would not be a local diffeomorphism, which is impossible in view of theorem 7.

Conclusion

The diversity of the approaches to supermanifold theory existing at present has not been given a unified treatment yet. Although the Berezin–Leĭtes–Kostant version [BerL, Ko, Ber2, L, BnL, Ma, BBH, BSV1] is probably the nearest one to a comprehensive viewpoint, we still believe that any other approach is also a contribution towards a thorough insight into the "true" notion of a supermanifold (and, thereby, a supergroup) rather than a "half-baked *ad hoc* definition" [BMF].

A wide-spread point of view (shared by us) is that the "naïve" theory with a non-trivial "ground algebra" of coefficients [DeW, Ro1, Ro2, Ro3, Ro4, Ba, BoG, HQR, JP2, CB1, CB2, RaC, CD, Rn, BBH, BBHP, VV, Kh, Br, KoN, MK] can be most probably rewritten in the language of relative categories of graded manifolds—namely, if M is a supermanifold over a graded-commutative algebra, Λ, of "supernumbers," then M can be identified with a superbundle $X \to \operatorname{Spec} \Lambda$, where X is a (possibly, infinite-dimensional [Mo, Scm]) Berezin–Leĭtes–Kostant supermanifold. This idea was repeatedly uttered by Leĭtes, mostly in private communications. Unfortunately, the fact is that almost no written evidence has been produced in support of the idea (apart from some initial work done by Penkov [P]), so its status remains that of a plausible conjecture (although backed by the prominence similar constructions have won in category theory and theory of topoi [J]) and by no means that of a mathematical result.

The well-known Batchelor's theorem [Ba], which is indeed one of the highlights of "supermathematics", states that every smooth finite-dimensional DeWitt supermanifold [DeW, CB1, CB2, RaC, CD, BBH] over a finite-dimensional Grassmann algebra of coefficients is an image under the change of base functor of a smooth finite-dimensional graded manifold.

However, it is very important to stress that the change of base functor is not an equivalence of categories—there are more morphisms between DeWitt supermanifolds than between the corresponding graded manifolds—one of the corollaries being the fact that the category of DeWitt supergroups is richer than that of Berezin–Leĭtes–Kostant supergroups. While one can still hope to describe DeWitt supergroups as deformations of Berezin–Leĭtes–Kostant supergroups over the base $\operatorname{Spec} \Lambda$ (in the spirit of, say, [FF]), the supergroups resulting from a more general class of supermanifolds—the Rogers supermanifolds [Ro1, Ro2, Ro3, Ro4, Ba, BoG, HQR, JP2, RaC, Rn, BBH, BBHP, VV, Kh, KoN, MK]—seemingly are not amenable to such a treatment.

The ultimate question arising in this connection is whether the Rogers supergroups are worthy of study. (Compare the points of view in, say, [Ro2, Ro3] and [Pi].) Indeed, it is known that any finite-dimensional Lie superalgebra comes from a supergroup which belongs at most to the DeWitt category (and thus falls within reach of deformation theory approach) [BerL, Ko, Ro2, Pe1, Pe3, Pe4].

Our conclusion is that the answer to the above question is a definite "yes", and the problem of rewording the "naïve" approach in the language of Berezin–Leĭtes–Kostant supermanifolds (possibly infinite dimensional) and their deformations becomes substantial.

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